Computing twisted conjugacy classes in free groups using nilpotent quotients

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Abstract

There currently exists no algebraic algorithm for computing twisted conjugacy classes in free groups. We propose a new technique for deciding twisted conjugacy relations using nilpotent quotients. Our technique is a generalization of the common abelianization method, but admits significantly greater rates of success. We present experimental results demonstrating the efficacy of the technique, and detail how it can be applied in the related settings of surface groups and doubly twisted conjugacy.

1 Introduction

Given two elements x and y in a group G and an endomorphism $\varphi : G \to G$, we say that x and y are *twisted conjugate* if there is some z such that

$$x = \varphi(z)yz^{-1}.$$

Twisted conjugacy is a generalization of the ordinary conjugacy relation in groups, and the computation of twisted conjugacy classes is a problem of considerable difficulty for many groups G.

Computing twisted conjugacy classes (also called "Reidmeister classes") is of interest in various algebraic contexts. Our own approach will be motivated by Nielsen fixed point theory, though other motivations exist (see [Bogopolski *et al.*, 2006], in which the twisted conjugacy problem in free groups arises naturally in the context of the ordinary conjugacy problem in certain other groups).

Extant algebraic techniques for computing twisted conjugacy classes are *ad hoc* in nature, and the goal of this paper is to present a new technique which is more generally applicable (though still not in general algorithmic), along with experimental results demonstrating its success.

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Our technique is an extension of the common abelianization technique. Viewing the abelianization as the first nilpotent quotient, we show that twisted conjugacy classes can be distinguished with increasing success rates when projected into nilpotent quotients of increasing nilpotency class. These projections, and the computations necessary to compute twisted conjugacy in nilpotent groups, can be fairly intensive, and as such we have implemented our technique in the MAGMA programming language.

In Section 2 we give the motivation for the twisted conjugacy problem from Nielsen theory, Section 3 is an outline of our technique, Section 4 gives some examples, Section 5 presents our experimental results, and Sections 6 and 7 show how the technique can be adapted into two natural generalizations of the main problem.

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2 Nielsen fixed point theory

Our principal motivation for studying the twisted conjugacy problem is Nielsen fixed point theory (standard references are [Jiang, 1983] and [Kiang, 1980]). Given a map $f : X \to X$ of a space with universal covering space $\widetilde{X}$ and projection map $p : \widetilde{X} \to X$, the fixed points of f are partitioned into classes of the form $p(\mathrm{Fix}(\widetilde{f}))$, where $\widetilde{f}$ is a lift of f. If a single lift $\widetilde{f}$ is fixed once and for all, each fixed point class can be expressed as $p(\mathrm{Fix}\,\alpha^{-1}\widetilde{f})$ for some $\alpha \in \pi_1(X)$. A central problem in Nielsen fixed point theory is to determine the number of "essential" fixed point classes of a mapping. This number gives a lower bound for the minimal number of fixed points for maps in the homotopy class of f, and in the case when X is a manifold of dimension at least 3, this "Nielsen number" is in fact equal to the minimal number of fixed points.

A fundamental question when counting fixed point classes is the following: given two elements $\alpha, \beta \in \pi_1(X)$, when is $p(\mathrm{Fix}(\alpha^{-1}\widetilde{f})) = p(\mathrm{Fix}(\beta^{-1}\widetilde{f}))$? The question can be answered with an elementary argument in covering space theory. Two such fixed point classes are equal if and only if there is some $\gamma \in \pi_1(X)$ with

$$\alpha = \varphi(\gamma)\beta\gamma^{-1},$$

where $\varphi : \pi_1(X) \to \pi_1(X)$ is the map induced by f on the fundamental group. If the above holds, we say that the elements α and β are *twisted conjugate* (if φ is the identity map, this is the ordinary conjugacy relation). The twisted conjugacy classes of f are also called *Reidemeister classes*, and the set of such classes is denoted $\mathcal{R}(\varphi)$.

In the case where f is a selfmap of a compact surface with boundary, the fundamental group $\pi_1(X)$ will be a free group, say $\pi_1(X) = \langle g_1, \ldots, g_k \rangle$. No

general algorithm is known for computing the Nielsen number of such a map (the case $k = 1$ is classically known, and an algorithm for the case $k = 2$ has recently been developed by Yi and Kim [Yi and Kim, 2008], extending work by Joyce Wagner [Wagner, 1999]). Fadell and Husseini, in [Fadell and Husseini, 1983], proved:

Theorem 2.1 (Fadell, Husseini, 1983). *Let φ be the map induced by f on $\pi_1(X)$. Then the Nielsen number of f is the number of terms with nonzero coefficient in*

$$RT(\varphi) = \rho \left(1 - \sum_{i=1}^{n} \frac{\partial}{\partial g_i} \varphi(g_i) \right),$$

where ∂ is the Fox Calculus operator (see [Crowell and Fox, 1963]), and $\rho : \mathbb{Z}\pi_1(X) \to \mathbb{Z}\mathcal{R}(\varphi)$ is the linear extension of the projection into twisted conjugacy classes.

The theorem above thus reduces the problem of computing the Nielsen number to the application of the projection ρ, which requires some algorithm for distinguishing twisted conjugacy classes. No such algorithm exists in the literature, though Bogopolski, Martino, Maslakova, and Ventura give an algorithm in [Bogopolski *et al.*, 2006] for the case where φ is an isomorphism. Their algorithm involves the traintracks machinery of Bestvina and Handel [Bestvina and Handel, 1992]. This paper explores a purely algebraic technique which can be implemented by existing computer algebra systems, and in some cases can be done by hand.

3 Abelian and nilpotent quotients

Throughout this and the next two sections, let G be a finitely generated free group, and let $\varphi : G \to G$ be an endomorphism. For an element $g \in G$, let $[g]$ denote the twisted conjugacy class of g. Our goal is to confirm or deny the equality $[g] = [h]$ for two group elements g and h.

Existing algebraic techniques for deciding equality of such classes are surveyed in [Hart, 2005]. One such technique is the algorithm of Wagner [Wagner, 1999], which is applicable if the mapping φ satisfies the combinatorial "remnant" condition (this condition is satisfied with probability approaching 1 as the word lengths of the images of generators increase). For general mappings (without remnant), distinguishing classes is typically done by projecting into the abelianization $\bar{G}$ and solving the twisted conjugacy relation there.

Abelianization is often sucessful at showing that two classes are distinct, but cannot be used to show that two classes are equal. Projecting into nilpotent quotients is a natural generalization of the abelianization technique, and we show how it can also furnish a technique for equating classes.

We will use the commutator notation $[x, y] = xyx^{-1}y^{-1}$. Let $\gamma_n(G)$ be the terms of the lower central series, and let $\widehat{G}_n = G/\gamma_n(G)$. Each of the groups $\widehat{G}_n$ are nilpotent of class n. The abelianization is of course $\bar{G} = \widehat{G}_1 =$

$G/\gamma_1(G)$. In general, we will use a bar to indicate projection of elements into the abelianization, and a hat to indicate projection into $\widehat{G}_n$, with the value of n to be understood by context.

Computation in $\bar{G}$ is made easy by commutativity. For $n > 1$, the groups $\widehat{G}_n$ are not commutative, but powerful commutation rules make computation possible. The following easily verifiable commutator identities hold in any group:

$$\begin{aligned} yx &= [y,x]xy, \\ [y,x] &= [x,y]^{-1}, \\ [xy,z] &= [x,[y,z]][y,z][x,z] \end{aligned}$$

These rules have a nicer form in a class 2 nilpotent group, where all commutators will freely commute:

Proposition 3.1. *If G is a class 2 nilpotent group, then, for any $x, y, z \in G$, we have*

$$yx = xy[x,y]^{-1}, \tag{1}$$

$$[y,x] = [x,y]^{-1}, \tag{2}$$

$$[xy,z] = [x,z][y,z]. \tag{3}$$

In a class 2 nilpotent group, we may use (1) to exchange the order of any non-commutator elements. Using (3) we may write any commutator as a product of commutators of generators. Having reduced all commutators to commutators of generators, we may use (2) to ensure that the generators appear in a prescribed order. Viewed as a set of word rewriting rules, Proposition 3.1 suggests that there will be some sort of normal form in nilpotent groups which can be used to compare words.

The desired normal form is provided by a theorem of P. Hall. Before stating the theorem, we give some terminology and notation, following [Hall, 1957].

For a free group G, consider the elements which can be formed by taking the closure of the generator set under the commutator operation. Of these elements, the generators are referred to as *weight 1 commutators*, and the weight of any non-generator element is defined to be the sum of the weights of the elements of which it is a commutator.

Hall showed that words in $\widehat{G}_n$ can be given in a normal form consisting of a product of certain *basic commutators* of weight n or less given in some proscribed order. The construction of the basic commutators is somewhat involved, and we refer to [Hall, 1957] and [Magnus *et al.*, 1976] for the details. For the purpose of the examples in this paper, it is sufficient to know that in a group of rank 2, say $G = \langle a, b \rangle$: the basic weight 1 commutators are a and b, and the only basic weight 2 commutator is $[a,b]$ (the commutator $[b,a]$ is not basic, since it is expressable as $[a,b]^{-1}$). We also refer to Theorem 5.11 of [Magnus *et al.*, 1976] which gives a combinatorial formula due to Witt for C_n, the number of basic weight n commutators:

$$C_n = \frac{1}{n} \sum_{d|n} \mu(d) k^{n/d},$$

where μ is the Möbius function, and k is the number of generators of G.

Theorem 3.2 (P. Hall, 1957)**.** *For any $x \in G$, we can write the projection $\widehat{x} \in \widehat{G}_n$ as*

$$\widehat{x} = \prod_i \widehat{c}_i^{k_i},$$

where $\{c_i\}$ is the sequence of basic commutators of weight less than or equal to n.

This form for $\widehat{x}$ is unique up to the ordering of the weight n basic commutators, and we call this the Hall normal form *for $\widehat{x}$.*

Since $\varphi(\gamma_n(G)) \subset \gamma_n(G)$, there is a well defined quotient mapping $\widehat{\varphi} : \widehat{G}_n \to \widehat{G}_n$. Thus it makes sense to ask, for $h, g \in G$, whether or not $[\widehat{h}] = [\widehat{g}]$ in $\widehat{G}_n$, that is, whether or not there is some $z \in \widehat{G}_n$ with

$$\widehat{h} = \widehat{\varphi}(z)\widehat{g}z^{-1}.$$

If $[\widehat{h}] \neq [\widehat{g}]$ in $\widehat{G}_n$, then we know that $[h] \neq [g]$ in G.

4 Some examples

We begin with a sample computation by hand, showing how the Hall normal form can be used to solve twisted conjugacy relations.

Example 4.1. We will compute the Nielsen number of the map on a surface with fundamental group $G = \langle a, b \rangle$ which induces the homomorphism:

$$\varphi : \begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & b^2a^4 \end{array}$$

Theorem 2.1 gives

$$RT(\varphi) = \rho(-1 - b)$$

and thus we need only decide whether or not $[1] = [b]$. First we attempt to equate these classes in the abelianization $\bar{G}$. Writing elements additively, an element $z \in \bar{G}$ is of the form $z = n\bar{a} + m\bar{b}$, and we wish to solve

$$0 = \bar{\varphi}(z) + \bar{b} - z.$$

We compute $\bar{\varphi}(z) = n(\bar{a} + \bar{b}) + m(4\bar{a} + 2\bar{b})$ and $-z = -n\bar{a} - m\bar{b}$, and the above equation becomes

$$-\bar{b} = 4m\bar{a} + (n + m)\bar{b},$$

and we can solve for n and m to find that $z = -\bar{a}$ is a solution.

Remark 4.2. Given that 1 and b are twisted conjugate in the abelianization by the element $-\bar{a}$, we might hope that 1 and b are twisted conjugate in the group G by the element a^{-1}. This is not the case, however, as $\varphi(a^{-1})ba = b^{-1}a^{-1}ba$. The possibility remains, however, that these elements are twisted conjugate by some more complicated word which abelianizes to $-\bar{a}$.

Having failed to decide the twisted conjugacy after a check in the abelianization, we proceed to the class 2 nilpotent quotient $\widehat{G}_2$. Any $z \in G_2$ is of the form $z = \widehat{a}^n \widehat{b}^m [\widehat{a}, \widehat{b}]^k$. We wish to solve

$$1 = \widehat{\varphi}(z)\widehat{b}z^{-1}.$$

We already know by the above calculation, however, that any such element z must abelianize to $\widehat{a} \in \widehat{G}$ in order to satisfy the twisted conjugacy equation. Thus we may assume that $z = \widehat{a}^{-1}[\widehat{a}, \widehat{b}]^k$. Now we compute

$$z^{-1} = [\widehat{a}, \widehat{b}]^{-k}\widehat{a} = \widehat{a}[\widehat{a}, \widehat{b}]^{-k},$$

and

$$\widehat{\varphi}(z) = \widehat{b}^{-1}\widehat{a}^{-1}[\widehat{a}\widehat{b}, \widehat{b}^2\widehat{a}^4]^k = \widehat{a}^{-1}\widehat{b}^{-1}[\widehat{a}^{-1}, \widehat{b}^{-1}][\widehat{a}, \widehat{b}]^{2k}[\widehat{b}, \widehat{a}]^{4k} = \widehat{a}^{-1}\widehat{b}^{-1}[\widehat{a}, \widehat{b}]^{-2k+1},$$

where we have used the rules of Proposition 3.1, together with the identity $[x^i, z] = [x, z]^i$, which follows from setting $x = y$ in identity (3).

We are now ready to test the twisted conjugacy equation above. The right hand side is:

$$\widehat{\varphi}(z)\widehat{b}z^{-1} = \widehat{a}^{-1}\widehat{b}^{-1}[\widehat{a}, \widehat{b}]^{-2k-8}\widehat{b}\widehat{a}[\widehat{a}, \widehat{b}]^{-k} = [\widehat{a}, \widehat{b}]^{-3k-8}.$$

Setting this equal to 1 requires that $-3k - 8 = 0$, which is impossible for $k \in \mathbb{Z}$. Thus there can be no such $z \in \widehat{G}_2$, and so $[1] \neq [b]$ and the Nielsen number is 2.

The example above involved a computation first in $\bar{G}$ and then in $\widehat{G}_2$. In each step, there are essentially two types of computational operations involved. The first is the term rewriting to obtain the Hall normal form, which was fairly easy in this case but in general can be quite tedious (though completely algorithmic). The second is finding the solution to a linear system, which was used to solve for n and m in the $\bar{G}$ step, and used to solve for k in the $\widehat{G}_2$ step (in a free group with more generators, this would have been a linear system with more than one equation).

As is to be expected, computation of twisted conjugacy classes in many cases may require checks in $\widehat{G}_n$ for $n > 2$. Such examples can easily be constructed by a computer search.

Example 4.3. Let $G = \langle a, b \rangle$, and let

$$\varphi : \begin{array}{ccc} a & \mapsto & aba^{-1} \\ b & \mapsto & a^{-2}b^4 \end{array}$$

Theorem 2.1 gives

$$RT(\varphi) = \rho(aba^{-1} - a^{-2} - a^{-2}b - a^{-2}b^2 - a^{-2}b^3).$$

It can be verified that all five of the above terms are twisted conjugate in $\widehat{G}_n$ for $n \in \{1, 2, 3\}$, but none are twisted conjugate in $\widehat{G}_4$. Thus the Nielsen number is 5.

We now turn to the question of verifying equality of twisted conjugacy classes. Applying the process described above to two words which are in fact twisted conjugate will result in a non-terminating sequence of computations in the groups $\widehat{G}_n$, each time resulting in solutions for the various elements above labeled z.

To avoid such an infinite computation, we propose a technique in the spirit of Remark 4.2. In the abelianization $\bar{G}$, if a solution element z is obtained, a sequence of "candidates" for testing twisted conjugacy in G is constructed by producing all possible reorderings of the generators appearing in z. Thus if we obtain an element $z = \bar{a} - 2\bar{b}$, our list of candidates will be

$$ab^{-2}, b^{-1}ab^{-1}, b^{-2}a.$$

Each of these elements can be tested by twisted conjugation as a candidate for realizing the twisted conjugacy in G.

A similar process can be carried out in $\widehat{G}_n$ for $n > 1$: after obtaining $z \in \widehat{G}_n$, our list of candidates is obtained by inserting the weight n basic commutators appearing in z in all possible orderings and in various forms into each of our candidates previously obtained in our check from $\widehat{G}_{n-1}$.

Example 4.4. Let $G = \langle a, b \rangle$, and

$$\varphi : \begin{array}{ccc} a & \mapsto & a^2ba \\ b & \mapsto & b^2a \end{array}$$

We will use the above candidates checking construction to show that $[a] = [a^2b]$ (the elements a^2b and a appear in $RT(\varphi)$).

Our check in $\bar{G}$ shows that the two elements are twisted conjugate by $-\bar{b}$. Thus our only candidate from $\bar{G}$ is the element b^{-1}, but a computation shows that

$$\varphi(b^{-1})(a^2b)b = a^{-1}b^{-2}a^2b^2 \neq a.$$

Now we do a check in $\widehat{G}_2$, and we find that the two elements are twisted conjugate by $\widehat{b}^{-1}[\widehat{a}, \widehat{b}]^{-1} \in \widehat{G}_2$. Now our list of candidates is:

$$\begin{aligned}
g_1 &= b^{-1}[a, b]^{-1} = ab^{-1}a^{-1} \\
g_2 &= b^{-1}[a^{-1}, b^{-1}]^{-1} = b^{-2}a^{-1}ba \\
g_3 &= b^{-1}[a^{-1}, b] = b^{-1}a^{-1}bab^{-1} \\
g_4 &= b^{-1}[a, b^{-1}] = b^{-1}ab^{-1}a^{-1}b \\
g_5 &= [a, b]^{-1}b^{-1} = bab^{-1}a^{-1}b^{-1} \\
g_6 &= [a^{-1}, b^{-1}]^{-1}b^{-1} = b^{-1}a^{-1}bab^{-1} \\
g_7 &= [a^{-1}, b]b^{-1} = a^{-1}bab^{-2} \\
g_8 &= [a, b^{-1}]b^{-1} = ab^{-1}a^{-1}
\end{aligned}$$

Computing each of $\varphi(g_i)a^2b(g_i)^{-1}$ reveals that $\varphi(g_1)a^2bg_1^{-1} = a$, and so $[a] = [a^2b]$.

Note that in the above example we inserted the basic commutator $[a,b]^{-1}$ into the word b^{-1} in each of two positions in one of four forms, these being the four versions of this commutator in G which become $[\widehat{a},\widehat{b}]^{-1}$ when projected into $\widehat{G}_2$. Constructing the various forms of a weight 3 commutator to use in a list of candidates would be cumbersome, and we do not attempt this construction for nilpotency class higher than 2.

As an alternative to the candidates construction procedure, a more pedestrian approach is always available: after a check for twisted conjugacy in $\widehat{G}_n$, use as list of candidates all words in G of word length n. This produces a different list of candidates from that described above, but is guaranteed to find an element realizing the twisted conjugacy if n is sufficiently high.

5 Success rates

The technique exhibited in the examples above can be summarized as follows: Starting with $n = 1$, write an expression for a generic element $z \in \widehat{G}_n$ and the element $\widehat{\varphi}(z)$ in Hall normal form. Solve a linear system to decide if the elements are twisted conjugate in $\widehat{G}_n$. If the elements are not twisted conjugate in $\widehat{G}_n$, then the elements are not twisted conjugate in G. If the elements are twisted conjugate in $\widehat{G}_n$ by a unique element z, then construct a finite list of candidates (either intelligently by using the structure of z or in a brute force manner by taking all words of length n) for testing twisted conjugacy in G. If all of these candidates fail, then increment n and repeat the above.

This technique will decide any given twisted conjugacy problem provided that the following statement is true: If $\varphi : G \to G$ is a map on a free group, and g and h are two elements of G which are not twisted conjugate in G, then there is some n for which g and h are not twisted conjugate in $\widehat{G}_n$. Thus any non-twisted-conjugate elements will be detected as such in $\widehat{G}_n$ for some n. Such a statement does not hold in general, though, as the following argument shows.

Proposition 5.1. *For any $\varphi : G \to G$, if $g, h \in G$ are words such that $\varphi^n(g) \in h\gamma_n(G)$ for all n, then $[\widehat{g}] = [\widehat{h}]$ in $\widehat{G}_n$ for all n.*

Proof. The proof is based on the fact that $[\varphi(x)] = [x]$ for any element x, since $\varphi(x) = \varphi(x)xx^{-1}$. Iteration of the map gives $[\varphi^n(x)] = [x]$ for any n.

Now for our element g, we have $\widehat{\varphi}^n(\widehat{g}) = \widehat{h}$ in $\widehat{G}_n$, and so in particular $[\widehat{\varphi}^n(\widehat{g})] = [\widehat{h}]$. But $[\widehat{\varphi}^n(\widehat{g})] = [\widehat{g}]$ by the above, and so we have $[\widehat{g}] = [\widehat{h}]$. ☐

We can use the above to build homomorphisms $\varphi : G \to G$ with the property that there are words $g, h \in G$ with $[g] \neq [h]$ but $[\widehat{g}] = [\widehat{h}]$ in every nilpotent quotient $\widehat{G}_n$.

Example 5.2. Let $G = \langle a, b \rangle$, and let $\varphi : G \to G$ be the map

$$\varphi : \begin{array}{ccc} a & \mapsto & [b,a] \\ b & \mapsto & a^{-1}b \end{array}$$

Now we have $\varphi(\gamma_n(G)) \subset \gamma_{n+1}(G)$ for $n > 0$, since φ replaces a with a weight 2 commutator and all commutators in G involve the element a. Since $\varphi(a) \in \gamma_1(G)$, we have $\varphi^n(a) \in \gamma_n(G)$ for all n, and thus that $[\widehat{a}] = [1]$ in $\widehat{G}_n$ for all n.

But φ is a mapping with remnant, and Wagner's algorithm can be used to show that in fact $[a] \neq [1]$ in G.

Though the above example shows that nilpotent quotients cannot be used directly to solve twisted conjugacy problems in all cases, the technique gives very good success rates in experimental testing. We have created an implementation[1] of the process in the computational algebra system MAGMA [Bosma *et al.*, 1994].

There are two ways that an implementation of this techinque can fail to decide any given twisted conjugacy problem. One type of failure is when the linear system which arises in the twisted conjugacy computation in $\widehat{G}_n$ has infinitely many solutions. In such a case the check in $\widehat{G}_{n+1}$ will require finding an integral solution of a polynomial system, which will in general be difficult. Such a failure can only occur when the coefficients in the linear system give a singular matrix, which we expect to occur relatively infrequently.

A second type of failure is that the implementation exhausts its resources in computing the required Hall normal forms in $\widehat{G}_n$. Even for groups of 2 or 3 generators, human computation of the Hall form in class 3 or 4 is barely feasible. Since the number of basic commutators of weight n grows exponentially in n we expect that any computer implementation could conceivably exhaust its resources before detecting that two given twisted conjugacy classes are indeed distinct. We expect this type of failure to occur increasingly in free groups with large numbers of generators.

Tables 1 and 2 give experimental results of application of the nilpotent quotients technique to 10,000 randomly generated mappings on the free group on k generators for $k = 2, 3, 4$. In each case, a number $l \in \{2, 3, 4, 5\}$ is chosen and a mapping is generated by assigning the image of each generator to be a randomly chosen word of length at most l. Theorem 2.1 is applied to give a list of group elements which must be divided into their twisted conjugacy classes. A sucessful computation is one which is able to decide the twisted conjugacy of these elements.

Table 1 gives the rates of each of the two types of failure above. A "matrix failure" is declared when the linear system computation results in infinitely many solutions. A "complexity failure" is declared when the nilpotency class reaches 5, as this is the level at which the computation of the Hall normal form becomes difficult for our implementation (run on a personal computer, current in 2005). Since a single mapping can trigger both types of error (computation of $RT(\varphi)$ in general requires several twisted conjugacy decisions), the row percentages may not sum to 100%. The column labeled "average depth" gives the average nilpotency class required to distinguish twisted conjugacy classes in these random mappings. A depth of n indicates that a check in $\widehat{G}_n$ was necessary. The column labeled σ gives the standard deviations of the depths.

[1] Available from the author's web site: `http://www.messiah.edu/~cstaecker`

k	l	Success	Matrix failure	Complexity failure	Avg. depth	σ
2	2	94.27%	4.30%	2.97%	1.10	0.28
	3	90.46%	6.66%	4.09%	1.13	0.34
	4	85.91%	7.75%	9.06%	1.18	0.42
	5	84.20%	9.89%	8.66%	1.17	0.38
3	2	87.45%	11.26%	4.38%	1.18	0.45
	3	85.60%	13.77%	4.74%	1.17	0.42
	4	85.20%	12.38%	6.19%	1.21	0.47
	5	84.46%	13.36%	6.29%	1.23	0.45
4	2	88.87%	10.95%	3.48%	1.13	0.41
	3	85.60%	13.77%	4.74%	1.16	0.42
	4	84.51%	14.56%	5.87%	1.21	0.45
	5	84.13%	15.04%	5.64%	1.21	0.42

Table 1: Results of testing for success rates on random mappings of word length l on the free group on k generators.

n	l	Nilpotent quotients	Abelianization	Wagner's Alg.
2	2	94.27%	82.75%	41.80%
	3	90.46%	70.45%	48.32%
	4	85.91%	59.07%	54.71%
	5	84.20%	51.14%	59.83%
3	2	90.98%	77.96%	15.54%
	3	87.45%	65.99%	24.70%
	4	85.20%	55.17%	33.72%
	5	84.46%	47.43%	41.45%
4	2	88.87%	76.50%	5.84%
	3	85.60%	64.76%	13.67%
	4	84.51%	54.63%	22.63%
	5	84.13%	47.81%	30.18%

Table 2: Comparison of success rates of various techniques on random mappings of word length l on the free group on k generators.

Table 2 gives our sucess rates compared to the existing techniques of abelianization and Wagner's algorithm. The technique used for data in the "Abelianization" column uses only the abelianization for distinguishing classes, and uses only the identity $[\varphi(g)] = [g]$ for equating classes (this is the general strategy employed in [Hart, 2005]). The column labeled "Wagner's alg." records the percentages of maps satisfying Wagner's remnant condition, for which her algorithm will apply (that the percentages grow in l is expected in light of Theorem 3.7 of [Wagner, 1999]).

6 Surfaces without boundary

The nilpotent quotients process can be used with minimal modifications when G is the fundamental group of a compact hyperbolic surface without boundary. Work of Fadell and Husseini in [Fadell and Husseini, 1983] together with a technique by Davey, Hart, and Trapp [Davey *et al.*, 1996] reduce the computation of the Nielsen number on compact hyperbolic surfaces without boundary to the computation of twisted conjugacy classes in the fundamental group.

Wagner's technique does not apply if G is not a free group, and neither will the techniques of [Bogopolski *et al.*, 2006]. Because surface groups are easily expressed in terms of commutator relations, we can use the nilpotent quotients technique with only trivial modifications in this setting.

The only modification that must be made is to the precise structure of the Hall normal form. For instance, if G is the fundamental group of the genus 2 compact surface, then G has group presentation $G = \langle a, b, c, d | [a, b][c, d] = 1\rangle$. The Hall normal form of an element of (e.g.) $\widehat{G}_2$ can be obtained by applying commutation rules as if G were the free group on 4 generators, along with an additional rule that $[c, d]^{-1} = [a, b]$. It is convenient for us that the group structure of G is so compatible with the Hall normal form.

The case of surfaces without boundary is somewhat more difficult to implement in Magma, as it involves computations in finitely-presented rather than free groups. The capabilities of Magma are somewhat lacking in this regard– in particular Magma (as of version 2.13-15) is unable to reliably solve the word problem in a surface group (although this word problem is solvable). This causes the candidates checking process to return false negatives, as the implementation may not recognize when two elements are actually equal. Statistics such as those in Table 1 are also difficult to produce in this setting as it is difficult to generate random endomorphisms of surface groups.

7 Doubly twisted conjugacy

We conclude with a brief discussion of how our technique can be applied to the *doubly twisted conjugacy* relation: Given two maps $\varphi, \psi : G \to H$ and two elements $h, k \in H$, we say that h and k are (doubly) twisted conjugate (we write

$[h] = [k]$) if there is an element $g \in G$ with

$$h = \varphi(g) k \psi(g)^{-1}.$$

This relation is fundamental in Nielsen coincidence theory (see [Gonçalves, 2005]), playing the same role as ordinary twisted conjugacy in fixed point theory.

For any n, the maps φ and ψ will induce maps $\widehat{\varphi}, \widehat{\psi} : \widehat{G}_n \to \widehat{H}_n$, and the doubly twisted conjugacy relation can in principle be solved by using Hall normal forms in $\widehat{G}_n$ and $\widehat{H}_n$ just as in the ordinary twisted conjugacy problem.

Example 7.1. Let $G = H = \langle a, b \rangle$, and let our maps be

$$\varphi : \begin{array}{ccc} a & \mapsto & b^2 a \\ b & \mapsto & a^{-2} \end{array} \qquad \psi : \begin{array}{ccc} a & \mapsto & a^3 \\ b & \mapsto & a^{-1} \end{array}$$

We will decide the twisted conjugacy of the elements b and b^{-1}.

We begin with check in the abelianization, where any element $z \in \bar{G}$ has the form $z = n\bar{a} + m\bar{b}$. We compute that $\bar{\varphi}(z) = (n - 2m)\bar{a} + 2n\bar{b}$ and $-\bar{\psi}(z) = (-3n + m)\bar{a}$, and thus we have

$$\bar{\varphi}(z) - \bar{b} - \bar{\psi}(z) = (-2n - m)\bar{a} + (2n - 1)\bar{b}.$$

Equating this with $\bar{b}$ and solving gives $n = 1$ and $m = -2$. This solution in the abelianization gives three candidates for twisted conjugacy:

$$ab^{-2}, b^{-1}ab^{-1}, b^{-2}a,$$

but checking each shows that none of these realize the twisted conjugacy in H.

We proceed to the class 2 nilpotent quotient, where any element $z \in \widehat{G}_2$ has the form $z = \widehat{a}^n \widehat{b}^m [\widehat{a}, \widehat{b}]^k$. Our computation in $\bar{G}$ shows that $n = 1$ and $m = -2$, simplifying our element to $z = \widehat{a}\widehat{b}^{-2}[\widehat{a}, \widehat{b}]^k$. We compute

$$\begin{aligned} \widehat{\varphi}(z) &= \widehat{b}^2 \widehat{a} (\widehat{a}^{-2})^{-2} [\widehat{b}^2 \widehat{a}, \widehat{a}^{-2}]^k = \widehat{a}^5 \widehat{b}^2 [\widehat{a}^5, \widehat{b}^2][\widehat{b}^2, \widehat{a}^{-2}]^k = \widehat{a}^5 \widehat{b}^2 [\widehat{a}, \widehat{b}]^{10+4k}, \\ \widehat{\psi}(z) &= \widehat{a}^3 (\widehat{a}^{-1})^{-2} [\widehat{a}^3, \widehat{a}^{-1}]^k = \widehat{a}^5, \end{aligned}$$

and so

$$\begin{aligned} \widehat{\varphi}(z) \widehat{b}^{-1} \widehat{\psi}(z)^{-1} &= \widehat{a}^5 \widehat{b}^2 [\widehat{a}, \widehat{b}]^{10+4k} \widehat{b}^{-1} \widehat{a}^{-5} = \widehat{a}^5 \widehat{b} \widehat{a}^{-5} [\widehat{a}, \widehat{b}]^{10+4k} \\ &= \widehat{b} [\widehat{a}^{-5}, \widehat{b}] [\widehat{a}, \widehat{b}]^{10+4k} = \widehat{b} [\widehat{a}, \widehat{b}]^{5+4k}. \end{aligned}$$

Equating this with $\widehat{b}$ gives $5 + 4k = 0$ which is impossible for integral k. Thus $[b] \neq [b^{-1}]$.

There is in the literature no analogue of Theorem 2.1 in coincidence theory, but presumably one may be available in the future, and our technique is currently the only available technique for distinguishing doubly twisted conjugacy classes (no version of Wagner's algorithm is known in coincidence theory, and

k_1	k_2	Success	Matrix failure	Complexity failure	Average depth	σ
2	2	92.38%	4.07%	3.55%	1.49	0.95
	3	98.97%	1.03%	0.%	1.09	0.28
	4	99.78%	0.22%	0.%	1.03	0.17
	5	99.90%	0.10%	0.%	1.01	0.12
3	2	30.53%	69.47%	0.%	1.	0.
	3	92.32%	5.83%	1.85%	1.41	0.80
	4	98.80%	1.20%	0.%	1.08	0.27
	5	99.71%	0.29%	0.%	1.03	0.18
4	2	14.56%	85.44%	0.%	1.	0.
	3	32.88%	67.12%	0.%	1.	0.
	4	91.98%	7.13%	0.89%	1.33	0.66
	5	98.50%	1.15%	0.%	1.08	0.27

Table 3: Success rates for doubly twisted conjugacy relations. Random mappings of word length 3 from the free group on k_1 generators to the free group on k_2 generators were tested in deciding twisted conjugacy between two random words of length at most 3.

the methods of [Bogopolski *et al.*, 2006] do not extend in an obvious way to doubly twisted conjugacy).

Table 3 gives success rates for the technique applied to 10,000 randomly generated twisted conjugacy relations. In each case, two random mappings of "word length" 3 (the quantity labeled l in Tables 1 and 2) are generated from the free group on k_1 generators to the free group on k_2 generators. Two random elements of the codomain group are generated with word length at most 3, and the implementation attempts to determine their twisted conjugacy. Entries in the table with no digits to the right of the decimal point are exact figures, e.g. in the case where $k_1 = 4$ and $k_2 = 3$ the depth was exactly 1 in each of the 10,000 test cases, and there were exactly 0 complexity failures.

Note that the technique is much less successful if the rank of the domain is greater than the rank of the codomain. This is to be expected, as the technique will fail when our linear system computation (always having more variables than equations if $k_1 > k_2$) yields infinitely many solutions. Note that such cases are not handled by our Magma implementation, but could in principle be done by hand. These would require finding integer solutions to polynomial systems, and so we do not always expect the computation to be successful, but particular examples may be computable.

Especially striking are the extremely high sucess rates when $k_2 > k_1$. The vast majority of these twisted conjugacy relations are decided in the abelianization (and with negative result), as the overdetermined linear systems are unlikely to have any solutions. For example in the case of $k_1 = 3$ and $k_2 = 5$, an equivalence in the abelianization would require our random elements to satisfy a linear system of 5 equations and 3 unknowns. If the elements are indeed equivalent

in the abelianization, a further equivalence in the class two nilpotent quotient would require our random data to satisfy a linear system of 10 equations and 3 unknowns (the free group on 5 generators has 10 basic weight 2 commutators, and the free group on 3 generators has 3 basic weight 2 commutators). This is not, of course, to say that twisted conjugate elements do not occur in such cases where $k_2 > k_1$, but only that this occurs very infrequently in the case of two words of length 3.

We have omitted from our testing the cases where k_1 or k_2 is 1. The twisted conjugacy relation is generally solvable in these cases by hand.

Let $\varphi, \psi : G \to H$ are maps of free groups where $G = \langle a \rangle$, and let $h, k \in H$ be two words. Then the twisted conjugacy problem is equivalent to finding some integer n with

$$1 = h^{-1}\varphi(a)^n k\psi(a)^{-n},$$

and this can typically be confirmed or denied by inspection.

If $\varphi, \psi : G \to H$ are maps of free groups and H has rank 1, let $G = \langle a_1, \ldots, a_n \rangle$, and note that $\varphi(\gamma_1(G)) \subset \gamma_1(H) = 1$ since H is abelian. Thus we have $\varphi(uv) = \varphi(vu[v,u]) = \varphi(vu)$ for any words $v, u \in G$, and similarly $\psi(uv) = \psi(vu)$. Thus, though there is no convenient normal form for elements $z \in G$, we can say in general that

$$\varphi(z) = \varphi(a_1^{m_1} \ldots a_n^{m_n}), \quad \psi(z) = \varphi(m_1^{k_1} \ldots m_n^{k_n})$$

by rearranging the generators of z.

Now to decide the twisted conjugacy of elements $h, k \in H$, we examine the equation

$$h = \varphi(a_1^{m_1} \ldots a_n^{m_n})k\psi(a_1^{-m_1} \ldots a_n^{-m_n}),$$

and we will be able to determine whether or not the above has solutions because it is an equation in the abelian group H. If the above has a solution, then the elements are twisted conjugate, and if not, they are not.

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